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On Pseudosimilarity in Trees

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Two vertices u and v in a graph G are said to be removal-similar if $G\backslash u \cong G\backslash v$. Vertices which are removal-similar but not similar are said to be pseudosimilar. A characterization theorem is presented for trees (later extended to forests and block graphs) with pseudosimilar vertices. It follows from this characterization that it is not possible to have three or more mutually pseudosimilar vertices in trees. Furthermore, removal-similarity combined with an extension of removal-similarity to include the removal of first neighbourhoods of vertices is sufficient to imply similarity in trees. Neither of these results holds, in general, if we replace trees by arbitrary graphs.

1. INTRODUCTION

Two vertices u and b in a graph G[1] are *similar*, denoted $u \sim_G v$ (or simply $u \sim v$ when G is clear from the context), if there exists an automorphism of G mapping u onto v. We are concerned, in this paper, with the notion of similarity and a related notion called pseudosimilarity among vertices in arbitrary trees.

An obvious consequence of the definition of similarity is that $u \sim_G v$ imples $G\backslash\{u\} \cong G\backslash\{v\}$,[2] which we abbreviate as $G\backslash u \cong G\backslash v$. According to

[1] We denote by $V(G)$ (resp. $E(G)$) the vertex (resp. edge) set of G.

[2] If $S \subseteq V(G)$, then $G\backslash S$ denotes the subgraph of G induced on $V(G)\backslash S$.

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Harary and Palmer [4], an incorrect proof of the celebrated Reconstruction Conjecture was based on the supposed truth of the converse, namely, that $G\backslash u \cong G\backslash v$ implies $u \sim_G v$. While this converse holds in certain interesting situations (e.g., in regular graphs) it is not true in general and counterexamples exist even among trees, the smallest of which is illustrated in Fig. 1. Two vertices u and v satisfying $G\backslash u \cong G\backslash v$ are said to be *removal-similar* in G. If, in addition, $u \not\sim_G v$, they are said to be *pseudosimilar*.[3]

The notion of pseudosimilarity has received considerable attention for both graphs and trees [1, 2, 4, 8]. Early work of Harary and Palmer [4] focused on pseudosimilarity in connected block graphs. Trees form the most interesting class of connected block graphs and, in Section 5, we show that there is no loss of generality in restricting the study of similarity and pseudosimilarity in block graphs to the special case of trees. Harary and Palmer's main result is an interesting characterization of pseudosimilar cutpoints in connected block graphs (equivalently, pseudosimilar vertices in trees). This characterization will be discussed in more detail in Section 3.

Harary and Palmer [4] present a general construction for graphs and trees with pseudosimilar vertices. In their construction graphs with pairs of pseudosimilar vertices are built from a number of disjoint copies of an arbitrary graph with dissimilar noncutpoints. In related work, Krishnamoorthy and Parthasarathy [9] construct a family of graphs with arbitrarily large sets of mutually pseudosimilar vertices.

A somewhat different characterization of graphs with pseudosimilar vertices has been provided recently by Godsil and Kocay [2]. They show that all finite graphs with pairs of pseudosimilar vertices can be constructed by a procedure originally presented in [7].

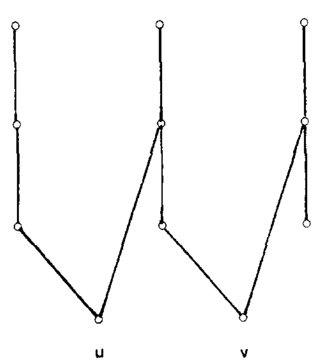


FIG. 1. Tree with pseudosimilar vertices u and v.

[3] Here we adopt the terminology introduced in [8]. The reader should be cautioned that this terminology differs from that used in other earlier papers (including [1]) in which pseudosimilarity referred to what we now call removal-similarity.

In other recent work, Kimble *et al.* [8] demonstrate an infinite family of graphs in which every vertex is pseudosimilar to some other vertex. Such a result is not possible for trees since, as shown by Harary and Palmer [4], trees do not admit pseudosimilar leaves. We are interested in exploring some of the other questions raised by the above results, in connection with trees. We present a new characterization of all trees with pseudosimilar vertices. It is a direct corollary of this characterization that a tree cannot have a set of more than two mutually pseudosimilar vertices.

In [1], the notion of pseudosimilarity and the results of Harary and Palmer [4] and Krishnamoorthy and Parthasarathy [9] are extended to k-pseudosimilarity and full k-pseudosimilarity. For a graph $G=(V, E)$, let Γ_v^k denote the set of vertices of distance less than or equal to k from vertex v in G (by definition $\Gamma_v^0=\{v\}$; Γ_v^1 is abbreviated Γ_v). Two vertices u and v are said to be *k-removal-similar* if $G\backslash\Gamma_u^k \cong G\backslash\Gamma_v^k$. Vertices u and v are *full k-removal-similar* if they are i-removal-similar for all $i \leqslant k$. In general, even full k-removal-similarity, for all k, does not imply similarity. In fact, there exist families of graphs with arbitrarily many pairwise full k-pseudosimilar (i.e., full k-removal-similar yet dissimilar) vertices, for all k [1].

In this paper, we restrict the study of k-removal-similarity to vertices of arbitrary trees. We show that, in contrast to the more general setting, full 1-removal-similarity is sufficient to imply similarity in trees. Similar results for edge removal-similarity are discussed in Section 7.

Section 2 introduces notation and definitions specific to this paper and presents some preliminary lemmas concerning the subtree structure of trees. Section 3 develops our characterization of pseudosimilarity in trees. This characterization is extended to arbitrary forests in Section 4. The tree characterization is further exploited in Section 5 to prove that full 1-removal-similarity is equivalent to similarity in trees. Sections 6 and 7 present further extensions (to block graphs) and related results on edge removal-similarity.

2. Trees and Branches

We will find it convenient to refer to rooted trees without always specifying the root. Our convention is that, unless otherwise specified, whenever some, possibly sub- or superscripted, upper case letter (e.g., X_i) denotes a rooted tree, then the corresponding lower case letter, with identical sub or superscripting, (e.g., x_i) denotes the root of that tree. When it becomes necessary to root an otherwise unrooted tree T at some vertex, say r, we will denote the resulting rooted tree (T, r).

If two rooted trees X and Y are isomorphic (that is, the isomorphism preserves the root) then we denote this by $X \overset{*}{\cong} Y$.

If $X_1,..., X_k$ are distinct rooted trees then we denote by $\langle X_1,..., X_k\rangle$ the (unrooted) tree with vertex set $\bigcup_{i=1}^{k} V(X_i)$ and edge set $\{(x_j, x_{j+1}) \mid 1 \leqslant j < k\} \cup \bigcup_{i=1}^{k} E(X_i)$. (Note that $\langle X_1, ..., X_k \rangle$ is indistinguishable from $\langle X_k, ..., X_1 \rangle$.) Graphically, if we represent the rooted tree X_i as in Fig. 2(a), then Fig. 2(b) denotes the tree $\langle X_1, ..., X_k \rangle$. The motivation for introducing this "chaining" of trees should be clear from the following proposition.

Let $d_T(u, v)$ denote the distance between vertices u and v in the tree T.

PROPOSITION 2.1. *If T is any tree with two specified vertices u and v, then there exist $s = d_T(u, v) + 1$ distinct rooted trees $X_1, ..., X_s$, such that $T = \langle X_1, ..., X_s \rangle$, $u = x_1$*, and $v = x_s$.

We now introduce a restricted type of rooted subtree that we call a branch. Branches allow us to describe, both qualitatively and quantitatively, the subtree structure of trees.

For each vertex v of a rooted tree (T, r) the rooted subtree (B, v), containing all vertices w such that v lies on the path from w to r in T is called a *branch* of (T, r). (If $v \neq r$, B is said to be a *proper* branch).

If T is an unrooted tree then the rooted tree B is said to be a *branch* of T if, for some rooting (T, r) of T, B is a branch of (T, r).

The following proposition is a straightforward consequence of the definition of a branch.

PROPOSITION 2.2. *If B is a branch of the rooted tree T, then there exist $r \geqslant 1$ rooted trees $Y_1, ..., Y_r$, such that*

(i) $B \stackrel{*}{\cong} Y_r$; *and*

(ii) $T = (\langle Y_1, ..., Y_r \rangle, y_1)$.

As Proposition 2.2 points out, a branch is not just a rooted subtree. In particular, if B is a branch of T then T with B removed is either empty or connected. This fact is exploited in the following lemma.

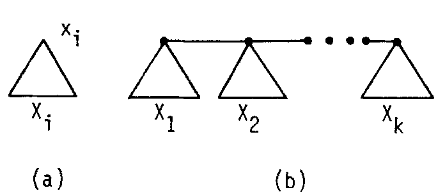


FIGURE 2

LEMMA 2.3. *If B is any branch of $\langle X, Y\rangle$, then either*

(i) *B is a branch of X;*

(ii) *B is a branch of Y;*

(iii) *X is a proper branch of B; or*

(iv) *Y is a proper branch of B.*

Proof. If both x and y belong to B, but neither X nor Y is a proper branch of B, then $\langle X, Y\rangle$ is disconnected by the removal of B, contrary to our assumptions. ∎

COROLLARY 2.4. *If $|X| \leqslant |Y|$, then x belongs to a unique branch of size $|X|$ in $\langle X, Y\rangle$, namely, X.*

LEMMA 2.5. *If B is any branch of $(\langle X_1,..., X_s\rangle, x_1)$, then either*

(i) *$B = (\langle X_i,..., X_s\rangle, x_i)$, where $1 \leqslant i \leqslant s$; or*

(ii) *B is a proper branch of X_i, for some i, $1 \leqslant i \leqslant s$.*

Proof. The result is a direct consequence of the definition of a branch. ∎

COROLLARY 2.6. *If $|X_i| \leqslant |X_s|$, $1 \leqslant i < s$, then $(\langle X_1,..., X_s\rangle, x_1)$ has a unique branch of size $|X_s|$, namely, X_s.*

LEMMA 2.7. *If $(\langle X_1, X_2\rangle, x_1) \stackrel{*}{\equiv} (\langle Y_1, Y_2\rangle, y_1)$ and $|X_2| \geqslant |Y_1|$, then $X_1 \stackrel{*}{\equiv} Y_1$ and $X_2 \stackrel{*}{\equiv} Y_2$.*

Proof. Let α be any isomorphism taking $(\langle X_1, X_2\rangle, x_1)$ onto $(\langle Y_1, Y_2\rangle, y_1)$. Then $\alpha(X_2)$ forms a proper branch of $(\langle Y_1, Y_2\rangle, y_1)$. By Lemma 2.5, this branch is either a proper branch of Y_1, contradicting the fact that $|X_2| \geqslant |Y_1|$, or is a branch of Y_2. Since $\alpha(x_2)$ is adjacent to y_1, it follows that $\alpha(X_1) = Y_1$ and $\alpha(X_2) = Y_2$. ∎

COROLLARY 2.8. *If $(\langle X_i,..., X_{s-1}\rangle, x_i) \stackrel{*}{\equiv} (\langle X_{j+1},..., X_s\rangle, x_{j+1})$, where $i \leqslant j < s$, then $X_k \stackrel{*}{\equiv} X_{k+j-i+1}$, $i \leqslant k < s-j+i-1$ and $X_s \stackrel{*}{\equiv} (\langle X_{s-j+i-1},..., X_{s-1}\rangle, x_{s-j+i-1})$.*

Proof. Straightforward induction on $s-j$. ∎

COROLLARY 2.9. *If $(\langle X_1,..., X_{s-1}\rangle, x_{s-1}) \stackrel{*}{\equiv} (\langle X_2,..., X_s\rangle, x_2)$, then $X_i \stackrel{*}{\equiv} X_{s-i+1}$, $1 \leqslant i \leqslant s$.*

Proof (*by induction on* s). If $s = 2$ or $s = 3$ the result is obvious. Suppose $s > 3$. Then, by Lemma 2.7, $X_{s-1} \stackrel{*}{\equiv} X_2$ and $(\langle X_1,..., X_{s-2}\rangle, x_{s-2}) \stackrel{*}{\equiv}$

$(\langle X_3,\ldots,X_s\rangle, x_3)$. By renaming $Y_1 = (\langle X_1, X_2\rangle, x_2)$, $Y_i = X_{i+1}$, $2 \leqslant i \leqslant s-3$, and $Y_{s-2} = (\langle X_{s-1}, X_s\rangle, x_{s-1})$, we have $(\langle Y_1,\ldots,Y_{s-3}\rangle, y_{s-3}) \stackrel{*}{\equiv} (\langle Y_2,\ldots,Y_{s-2}\rangle, y_2)$. Hence, by the induction hypothesis, $Y_i \stackrel{*}{\equiv} Y_{s-i-1}$ from which it follows that $X_i \stackrel{*}{\equiv} X_{s-i+1}$, $1 \leqslant i \leqslant s$. ∎

COROLLARY 2.10. *If* $(\langle X_1,\ldots,X_s\rangle, x_s) \stackrel{*}{\equiv} (\langle X_1,\ldots,X_s\rangle, x_1)$ *then* $X_i \stackrel{*}{\equiv} X_{s-i+1}$, $1 \leqslant i \leqslant s$.

Proof. Straightforward induction on s. ∎

We denote by $\mathrm{Br}\{X; T; v\}$ the number of (not necessarily disjoint) X-branches containing vertex v in the (possibly rooted) tree T. $\mathrm{Br}\{X; T\}$ denotes the number of (not necessarily disjoint) X-branches in T. Obviously, if $T_1 \cong T_2$ (or $T_1 \stackrel{*}{\equiv} T_2$, in the case of rooted trees) then $\mathrm{Br}\{X; T_1\} = \mathrm{Br}\{X; T_2\}$. The following lemma allows us to relate the branch structure of certain trees to the branch structure of their subtrees.

LEMMA 2.11. *If X, Y, and Z are rooted trees satisfying* $|X| \leqslant |Z| \leqslant |Y|$, *then*

$$\mathrm{Br}\{Z; \langle X, Y\rangle\} = \mathrm{Br}\{Z; \langle X, Y\rangle; x\} + \mathrm{Br}\{Z; Y\}.$$

Proof. Immediate from Lemma 2.3. ∎

LEMMA 2.12. *If B and (T, r) are rooted trees satisfying* $|B| < |T|$, *then*

$$\mathrm{Br}\{B; (T, r)\} = \mathrm{Br}\{B; T\} - \mathrm{Br}\{B; T; r\}.$$

Proof. It follows from Proposition 2.2 that the only branch of (T, r) containing r is (T, r) itself. Hence, if $|B| < |T|$, any B-branch of T containing r is not a branch of (T, r). ∎

3. A CHARACTERIZATION OF PSEUDOSIMILARITY IN TREES

We are now prepared to develop a new characterization of trees with pseudosimilar vertices. Before doing so let us recall the characterization presented by Harary and Palmer [4] expressed in our notation.

THEOREM A [4, Theorem 5]. *If T is any tree with pseudosimilar vertices u and v, then either*

(i) *for some integer $t > 2$, there exist rooted trees Y_k, $1 \leqslant k \leqslant 3t-1$, where $Y_k \stackrel{*}{\equiv} Y_{k+t}$, $1 \leqslant k \leqslant 2t-1$, such that $T = \langle Y_1,\ldots,Y_{3t-1}\rangle$, $u = y_t$, and $v = y_{2t}$; or*

(ii) *there exists a vertex* w *in the component* T' *of* $T\backslash u$ *containing* v *such that* w *and* v *are pseudosimilar in* T'.

Theorem A provides a quite explicit characterization of minimal trees with pseudosimilar vertices. An obvious question is whether a similar characterization holds for all trees with pseudosimilar vertices.

Recalling Proposition 2.1, it is easy to confirm

PROPOSITION 3.1. *If* T *is any tree with distinct removal-similar vertices* u *and* v, *then there exist* $s = d_T(u, v) + 1$ *distinct rooted trees* $X_1, \ldots, X_s$, *such that* $T = \langle X_1, \ldots, X_s \rangle$, $u = x_1$, $v = x_s$, $\langle X_1, \ldots, X_{s-1} \rangle \cong \langle X_2, \ldots, X_s \rangle$, *and* $X_1 \backslash x_1 \cong X_s \backslash x_s$.

For the remainder of this section let $T = \langle X_1, \ldots, X_s \rangle$, $P = \langle X_1, \ldots, X_{s-1} \rangle$, $Q = \langle X_2, \ldots, X_s \rangle$, and $R = \langle X_2, \ldots, X_{s-1} \rangle$. Note that $T = \langle X_1, (Q, x_2) \rangle = \langle (P, x_{s-1}), X_s \rangle$, $P = \langle X_1, (R, x_2) \rangle$, and $Q = \langle (R, x_{s-1}), X_s \rangle$. Obviously $|P| = |Q|$ if and only if $|X_1| = |X_s|$.

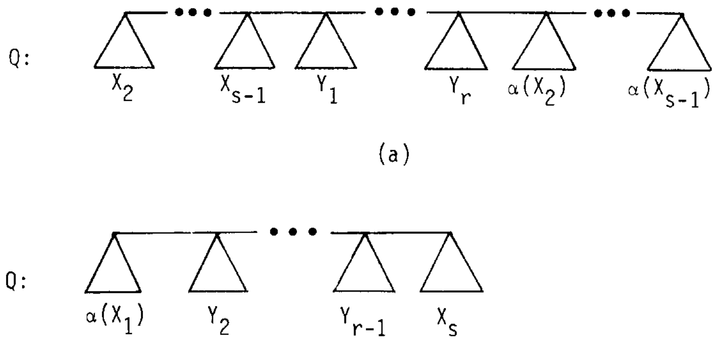
LEMMA 3.2. $P \cong Q$ *and* $X_1 \stackrel{*}{\cong} X_s$ *if and only if* $x_1 \sim_T x_s$.

Proof. If $x_1 \sim_T x_s$ then it follows from Corollary 2.10 that $P \cong Q$ and $X_1 \stackrel{*}{\cong} X_s$.

Conversely, suppose $P \cong Q$ and $X_1 \stackrel{*}{\cong} X_s$. If $s \leqslant 3$, then the fact that $x_1 \sim_T x_s$ is immediate. For $s > 3$, we proceed by induction on $|T|$, assuming that the hypothesis is true for all trees smaller than T. Let α be any isomorphism taking P onto Q. $\alpha(X_i)$ (resp. $\alpha(x_i)$) denotes the image of X_i (resp. x_i) under α. (It is assumed that $\alpha(X_i)$ is rooted at $\alpha(x_i)$). Suppose that $\alpha(x_1) \neq x_s$; otherwise there is nothing to prove. There are two cases.

(i) $\alpha(x_1) \in X_s \backslash x_s$. Since $\alpha(X_1) \not\subseteq X_s \backslash x_s$ (because $|X_1| = |X_s|$), it follows from Lemma 2.3 that (R, x_{s-1}) is a proper branch of $\alpha(X_1)$ and $(\alpha(R), \alpha(x_2))$ is a proper branch of X_s. Hence, by Proposition 2.2, there exist $r \geqslant 1$ rooted trees $Y_1, \ldots, Y_r$, such that $X_s = (\langle Y_1, \ldots, Y_r, (\alpha(R), \alpha(x_2)) \rangle, y_1)$ and $\alpha(X_1) = (\langle (R, x_{s-1}), Y_1, \ldots, Y_r \rangle, y_r)$(see Fig. 3(a)). Since $X_1 \stackrel{*}{\cong} X_s$, it follows by Corollary 2.9, that $(\alpha(R), \alpha(x_2)) \stackrel{*}{\cong} (R, x_{s-1})$, and hence $x_2 \sim_R x_{s-1}$ and $x_1 \sim_T x_s$.

(ii) $\alpha(x_1) \in X_i$, where $2 \leqslant i < s$. By Lemma 2.3, we know that $\alpha(X_1)$ is a branch of (R, x_{s-1}) and X_s is a branch of $(\alpha(R), \alpha(x_2))$. Hence, by Proposition 2.2, there exist $r \geqslant 2$ rooted trees $Y_1, \ldots, Y_r$, such that $(R, x_{s-1}) = (\langle Y_1, \ldots, Y_{r-1} \rangle, y_{r-1})$, $(\alpha(R), \alpha(x_2)) = (\langle Y_2, \ldots, Y_r \rangle, y_2)$, $\alpha(X_1) \stackrel{*}{\cong} Y_1$, and $X_s \stackrel{*}{\cong} Y_r$ (see Fig. 3(b)). It follows by our induction hypothesis, that y_1 and y_r are similar in $\langle Y_1, \ldots, Y_r \rangle$. Hence, $(R, x_{s-1}) \stackrel{*}{\cong} (\alpha(R), \alpha(x_2))$ or $x_2 \sim_R x_{s-1}$, from which it follows that $x_1 \sim_T x_s$. ∎



(a)

(b)

FIGURE 3

Let T be an arbitrary tree and let L denote the set of leaves of T. We call the leaves of $T \backslash L$ the *near-leaves* of T.

COROLLARY 3.3.

(i) [4, Theorem 4] *If T is any tree with removal-similar leaves u and v, then $u \sim_T v$.*

(ii) *If T is any tree with removal-similar near-leaves u and v, then $u \sim_T v$.*

It follows from Corollary 3.3 that, unlike the situation for general graphs [8], it is impossible to construct trees in which every vertex is pseudosimilar to some other vertex.

LEMMA 3.4. *If $P \cong Q$* and *$x_1 \not\sim_T x_s$ then $|X_j| \leqslant |X_1|$, $1 < j < s$.*

Proof. Suppose to the contrary that $|X_j| \leqslant |X_1|$, $1 \leqslant j < p < s$, and $|X_p| > |X_1|$. Let $Y = \langle X_1, ..., X_{p-1} \rangle$. Since $T = \langle X_1, (Q, x_2) \rangle$ it follows, by Lemma 2.11, that $\mathrm{Br}\{(Y, x_{p-1}); T\} = \mathrm{Br}\{(Y, x_{p-1});\ T;\ x_1\} + \mathrm{Br}\{(Y, x_{p-1}); (Q, x_2)\}$. Similarly, since $T = \langle (P, x_{s-1}), X_s \rangle$ it follows, by Lemma 2.11, that $\mathrm{Br}\{(Y, x_{p-1}); T\} = \mathrm{Br}\{(Y, x_{p-1});\ T;\ x_s\} + \mathrm{Br}\{(Y, x_{p-1});\ (P, x_{s-1})\}$. But clearly $\mathrm{Br}\{(Y, x_{p-1});\ T;\ x_1\} > 0$ and hence $\mathrm{Br}\{(Y, x_{p-1});\ T;\ x_s\} > 0$ or $\mathrm{Br}\{(Y, x_{p-1});\ (P, x_{s-1})\} > \mathrm{Br}\{(Y, x_{p-1});\ (Q, x_2)\}$. We consider the two cases separately.

(i) $\mathrm{Br}\{(Y, x_{p-1});\ (P, x_{s-1})\} > \mathrm{Br}\{(Y, x_{p-1});\ (Q, x_2)\}$. Since $\mathrm{Br}\{(Y, x_{p-1}); P\} = \mathrm{Br}\{(Y, x_{p-1}); Q\}$, it follows, by Lemma 2.12, that $\mathrm{Br}\{(Y, x_{p-1});\ Q;\ x_2\} > \mathrm{Br}\{(Y, x_{p-1});\ P;\ x_{s-1}\} \geqslant 0$. Consider any (Y, x_{p-1})-branch B in Q containing the vertex x_2. B must also contain the vertex x_p, since otherwise B is a branch of $\langle X_2,\dots,X_{p-1}\rangle$, contradicting the fact that $|Y|>|\langle X_2,\dots,X_{p-1}\rangle|$. Similarly B must contain all of X_j, $2\leqslant j\leqslant p$, since otherwise

$$|B|>|Q\backslash V(X_j)|,$$

by Lemma 2.3

$$\geqslant |P|-|X_p|$$

$$\geqslant |Y|.$$

Hence $|B|\geqslant |X_2|+\cdots+|X_p|$, contradicting our assumption that $|X_p|>|X_1|$.

(ii) $\mathrm{Br}\{(Y,x_{p-1});T;x_s\}>0$. Consider any (Y,x_{p-1})-branch B in T containing the vertex x_s. X_s must be a branch of B; otherwise, by Lemma 2.3, B contains $\langle X_1,\dots,X_{s-1}\rangle$, contradicting the fact that $|Y|<|X_1|+\cdots+|X_{s-1}|$. But, since $|X_j|\leqslant|X_1|$, $1<j<p$, it follows, by Corollary 2.6, that B (equivalently, (Y,x_{p-1})) contains a unique branch of size $|X_1|$, namely X_1 itself. Thus $X_1\stackrel{*}{\cong}X_s$ and, by Lemma 3.2, $x_1\sim_T x_s$, contradicting our assumptions.

Since both cases lead to contradictions, it follows that $|X_j|\leqslant|X_1|$, $1<j<s$. ∎

LEMMA 3.5. *If $P\cong Q$ and $x_1\not\sim_T x_s$ then for some integers $t>1$ and $h=2t+s-2$ there exist rooted trees Y_k, $1\leqslant k\leqslant h$, where $Y_i\stackrel{*}{\cong}Y_{i+t}$, $1\leqslant i\leqslant h-t$, such that $T=\langle Y_1,Y_2,\dots,Y_h\rangle$, $x_1=y_t$, and $x_s=y_{h-t+1}$.*

Proof. Let α be any automorphism taking P onto Q. We consider three cases.

(i) $\alpha(x_1)\in X_j\backslash x_j$, where $2\leqslant j<s$. By Lemma 2.3, either $\alpha(X_1)$ is a proper branch of X_j (impossible, by Lemma 3.4) or $\alpha(X_1)$ contains X_s as a proper branch (which contradicts the fact that $|X_1|=|X_s|$).

(ii) $\alpha(x_1)=x_j$, where $2\leqslant j<s$. Since $|X_1|=|X_s|$, it follows that $\alpha(X_1)=(\langle X_2,\dots,X_j\rangle,x_j)$ and $(\langle\alpha(X_2),\dots,\alpha(X_{s-1})\rangle,\alpha(x_2))=(\langle X_{j+1},\dots,X_s\rangle,x_{j+1})$. Hence, by Corollary 2.8, $X_k\stackrel{*}{\cong}X_{k+j-1}$, $2\leqslant k<s-j+1$, and $X_s\stackrel{*}{\cong}(\langle X_{s-j+1},\dots,X_{s-1}\rangle,x_{s-j+1})$. If $j=2$ then $x_1\sim_T x_s$, contradicting the hypothesis. If $j>2$, the lemma holds with $t=j-1$, $h=2t+s-2$ and $Y_i\stackrel{*}{\cong}X_{i+1}$, $1\leqslant i\leqslant t$.

(iii) $\alpha(x_1)\in X_s$. If $s=2$ then, by Proposition 2.1, it follows that there exist $r\geqslant 1$ rooted trees $Z_1,\dots,Z_r$ such that $\alpha(X_1)=(\langle Z_1,\dots,Z_r\rangle,z_r)$ and $X_s=(\langle Z_1,\dots,Z_r\rangle,z_1)$. More generally if $s>2$ then, since $|X_1|=|X_s|$ yet $X_1\stackrel{*}{\ncong}X_s$ (by Lemma 3.2), it follows, by Lemma 2.3, that $(\langle X_2,\dots,X_{s-1}\rangle,x_{s-1})$ is a proper branch of $\alpha(X_1)$. Thus, by Proposition 2.2, there exist $r\geqslant 1$ rooted trees $Z_1,\dots, Z_r$ such that $\alpha(X_1) = (\langle X_2,\dots, X_{s-1}, Z_1,\dots, Z_r\rangle, Z_r)$ and $X_s = (\langle Z_1,\dots, Z_r, \alpha(X_2),\dots, \alpha(X_{s-1})\rangle, z_1)$. Again, if $s = 2$ and $r = 1$ then $x_1 \sim_T x_S$, contradicting the hypothesis. If $r + s > 3$ then the lemma holds with $t = r + s - 2$, $h = 2t + s - 2$, and

$$\begin{aligned} Y_i &\overset{*}{=} X_{i+1}, && 1 \leqslant i < s-1, \\ &\overset{*}{=} Z_{i-s+2}, && s-1 \leqslant i < s+r-1. \quad \blacksquare \end{aligned}$$

An immediate consequence of Lemma 3.5 is that Lemma 3.4 can be strengthened as

COROLLARY 3.6. *If* $P \cong Q$ *and* $x_1 \not\sim_T x_s$, *then* $|X_1| = |X_s| > |X_j|$, $1 < j < s$.

Lemma 3.5 is extended to a characterization of trees with pseudosimilar vertices in

THEOREM 3.7. *If T is any tree with pseudosimilar vertices u and v, then for some integers $t > 1$ and $h = 2t + d_T(u, v) - 1$ there exist rooted trees Y_k, $1 \leqslant k \leqslant h$, where $Y_k \overset{*}{=} Y_{k+t}$, $1 \leqslant k \leqslant h - t$, such that $T = \langle Y_1, Y_2,\dots, Y_h\rangle$, $u = y_t$, $v = y_{h-t+1}$, and $\langle Y_1, Y_2,\dots, Y_t\rangle\backslash y_t \cong \langle Y_{h-t+1},\dots, Y_h\rangle\backslash y_{h-t+1}$.*

Proof. Immediate from Proposition 3.1 and Lemma 3.5. ∎

Another direct consequence of Lemma 3.5 (more specifically, Corollary 3.6) is that, unlike the case for general graphs, any set of mutually pseudosimilar vertices of a tree has cardinality at most two. Specifically,

THEOREM 3.8. *Let T be any tree with vertices u, v, and w. If u and v are pseudosimilar in T and u and w are pseudosimilar in T, then $v = w$.*

Proof. By Proposition 3.1 and Corollary 3.6, it follows that there exist $s = d_T(u, v) + 1$ distinct rooted trees $X_1,\dots, X_s$ such that $T = \langle X_1,\dots, X_s\rangle$, $u = x_1$, $v = x_s$, and $|X_1| = |X_s| > |X_j|$, $1 < j < s$. Similarly, there exist $r = d_T(u, w) + 1$ distinct rooted trees $Y_1,\dots, Y_r$ such that $T = \langle Y_1,\dots, Y_r\rangle$, $u = y_1$, $w = y_r$, and $|Y_1| = |Y_r| > |Y_j|$, $1 < j < r$. By Lemma 2.7, $X_1 \overset{*}{=} Y_1$ and hence $|X_s| = |Y_r|$. But, by Corollary 2.6, (T, u) has a unique branch of size $|X_s|$ namely, X_s. Hence $X_s = Y_r$ and, in particular, $v = w$. ∎

4. PSEUDOSIMILARITY IN FORESTS

Suppose F is any forest with pseudosimilar vertices u and v. If u and v belong to the same component T of F then it should be clear that u and v are pseudosimilar in T and hence the characterization of the previous section

(Theorem 3.7) holds. What if u and v belong to distinct components T_1 and T_2 in F? This turns out to be possible only when $T_1 \cong T_2$, say $\alpha(T_1) = T_2$, and $\alpha(u)$ and v are pseudosimilar in T_2.

LEMMA 4.1. *If T_1 and T_2 are distinct trees with $u \in V(T_1)$ and $v \in V(T_2)$, then u and v are (removal) similar in $T_1 \cup T_2$ if and only if u and v are (removal) similar in the tree $T_1 \cup T_2 \cup \{(u, v)\}$.*

Proof. The result for removal-similarity is obvious since $(T_1 \cup T_2)\backslash u \equiv (T_1 \cup T_2 \cup \{(u, v)\})\backslash u$ and $(T_1 \cup T_2)\backslash v \equiv (T_1 \cup T_2 \cup \{(u, v)\})\backslash v$. If u and v are similar in $T_1 \cup T_2$, then $(T_1, u) \overset{*}{\cong} (T_2, v)$. If α is any isomorphism taking (T_1, u) onto (T_2, v) then the automorphism δ given by

$$\begin{aligned} \delta(x) &= \alpha(x) && \text{if} \quad x \in V(T_1), \\ &= \alpha^{-1}(x) && \text{if} \quad x \in V(T_2), \end{aligned}$$

interchanges u and v in $T_1 \cup T_2$ (and hence also in $T_1 \cup T_2 \cup \{(u, v)\}$. If u and v are similar in $T_1 \cup T_2 \cup \{(u, v)\}$ then there is an automorphism exchanging them (see, for example, Corollary 1 of [4]), and hence $(T_1, u) \overset{*}{\cong} (T_2, v)$ and $u \sim_{T_1 \cup T_2} v$. ∎

THEOREM 4.2. *If F is any forest with pseudosimilar vertices u and v, where u and v belong to components T_1 and T_2, respectively, then there exists a vertex $w \in V(T_2)$ such that $(T_1, u) \overset{*}{\cong} (T_2, w)$ and v and w are pseudosimilar in T_2.*

Proof. If $T_1 = T_2$ the result is obvious. Otherwise, we know, by Lemma 4.1, that u and v are pseudosimilar in $T_1 \cup T_2 \cup \{(u, v)\}$. It follows, by Theorem 3.7, that for some $t > 1$ there exist rooted trees Y_i, $1 \leqslant i \leqslant 2t$, where $Y_i \overset{*}{\cong} Y_{i+t}$, $1 \leqslant i \leqslant t$, such that $(T_1, u) = (\langle Y_1, ..., Y_t \rangle, y_t)$ and $(T_2, v) = (\langle Y_{t+1}, ..., Y_{2t} \rangle, y_{t+1})$. Choosing $w = y_{2t}$, the result follows directly. ∎

5. FULL k-REMOVAL-SIMILARITY IN TREES

In Section 3, we presented a new characterization of trees with pseudosimilar vertices. It is natural to ask if this characterization can be extended to the notion of full k-pseudosimilarity (cf. Section 1). Surprisingly perhaps, this characterization is very simple since for $k = 1$ (and hence for all $k \geqslant 1$) full k-removal-similarity is equivalent to similarity in trees.

THEOREM 5.1. *If $T\backslash u \cong T\backslash v$ and $T\backslash \Gamma_u \cong T\backslash \Gamma_v$, then $u \sim_T v$.*

Proof. Suppose not. Let T be any smallest counterexample. Since

$T\backslash u \cong T\backslash v$, it follows, by Theorem 3.7, that there exist integers $t > 1$ and $h \geqslant 2t$ and rooted trees Y_k, $1 \leqslant k \leqslant h$, where $Y_k \overset{*}{=} Y_{k+t}$, $1 \leqslant k \leqslant h - t$, such that $T = \langle Y_1, ..., Y_h \rangle$, $u = y_t$ and $v = y_{h-t+1}$. We consider two cases:

(i) u and v are adjacent in T; that is $h = 2t$. Since $T\backslash u \cong T\backslash v$, it follows that $\langle Y_1, ..., Y_t \rangle \backslash y_t \cong \langle Y_{t+1}, ..., Y_{2t} \rangle \backslash y_{t+1}$. Furthermore, since $T\backslash \Gamma_u = (\langle Y_1, ..., Y_t \rangle \backslash \Gamma_{y_t}) \cup (\langle Y_{t+1}, ..., Y_{2t} \rangle \backslash y_{t+1})$, $T\backslash \Gamma_v = (\langle Y_1, ..., Y_t \rangle \backslash y_t) \cup (\langle Y_{t+1}, ..., Y_{2t} \rangle \backslash \Gamma_{y_{t+1}})$, and $T\backslash \Gamma_u \cong T\backslash \Gamma_v$, it follows that $\langle Y_1, ..., Y_t \rangle \backslash \Gamma_{y_t} \cong \langle Y_{t+1}, ..., Y_{2t} \rangle \backslash \Gamma_{y_{t+1}}$. Thus, y_1 and y_t are full 1-removal-similar in $\langle Y_1, ..., Y_t \rangle$ and so, by our minimality assumption, y_1 and y_t are similar in $\langle Y_1, ..., Y_t \rangle$. Hence $(\langle Y_1, ..., Y_t \rangle, y_t) \overset{*}{=} (\langle Y_{t+1}, ..., Y_{2t} \rangle, y_{t+1})$ and $u \sim_T v$, contradicting our assumption.

(ii) u and v are not adjacent in T; that is $h > 2t$. Let $A = \langle Y_1, ..., Y_{h-t} \rangle$. Since $T\backslash \Gamma_u \cong T\backslash \Gamma_v$ it follows that $\langle Y_1, ..., Y_{h-t-1} \rangle \cong \langle Y_{t+2}, ..., Y_h \rangle$ and hence $\langle Y_1, ..., Y_{h-t-1} \rangle \cong \langle Y_2, ..., Y_{h-t} \rangle$. Since $Y_1 \overset{*}{=} Y_{t+1}$, it follows (by Lemma 3.2 for $h = 2t + 1$ and Corollary 3.6 for $h > 2t + 1$) that $y_1 \sim_A y_{h-t}$. Thus, by Corollary 2.10, $Y_i \cong Y_{h-t-i+1}$, $1 \leqslant i \leqslant h - t$. Hence $Y_i \cong Y_{h-i+1}$, $1 \leqslant i \leqslant h$, from which it follows immediately that $u \sim_T v$, contradicting our assumption. ∎

6. Pseudosimilarity in Block Graphs

As was mentioned in the introduction, Harary and Palmer [4] developed their characterization of pseudosimilar vertices in the apparently more general context of connected block graphs. However, as the results of this section demonstrate, with respect to questions of similarity and pseudosimilarity, connected block graphs and their special case, trees, are essentially equivalent. A *block graph* is perhaps most easily defined as a graph in which each block (i.e., maximal biconnected component[4]) is complete [3]. For the remainder of this section, let G denote an arbitrary connected block graph with vertex set V and edge set E. Let B denote the set of blocks of G and C the set of cutpoints of G.

Harary and Palmer [4] exploit a tree description of G, called the *block-cutpoint-tree* [6] and denoted $T(G)$. $T(G)$ has vertex set $B \cup C$ and edge set $\{(b, c) \in B \times C \mid c \in b\}$ (see Fig. 4b). Obviously, $T(G)$ does not uniquely represent G in general. However, a slight extension of $T(G)$, which we call the *block-vertex-incidence-tree* of G (denoted $BV(G)$), is easily seen to provide a unique tree representation of G. $BV(G)$ is the tree with vertex set $B \cup V$ and edge set $\{(b, v) \in B \times V \mid v \in b\}$ (see Fig. 4c). If H is an arbitrary block graph with connected components $H_1, ..., H_t$, then $BV(H) = \bigcup_{i=1}^{t} BV(H_i)$.

[4] K_2 is considered to be biconnected.

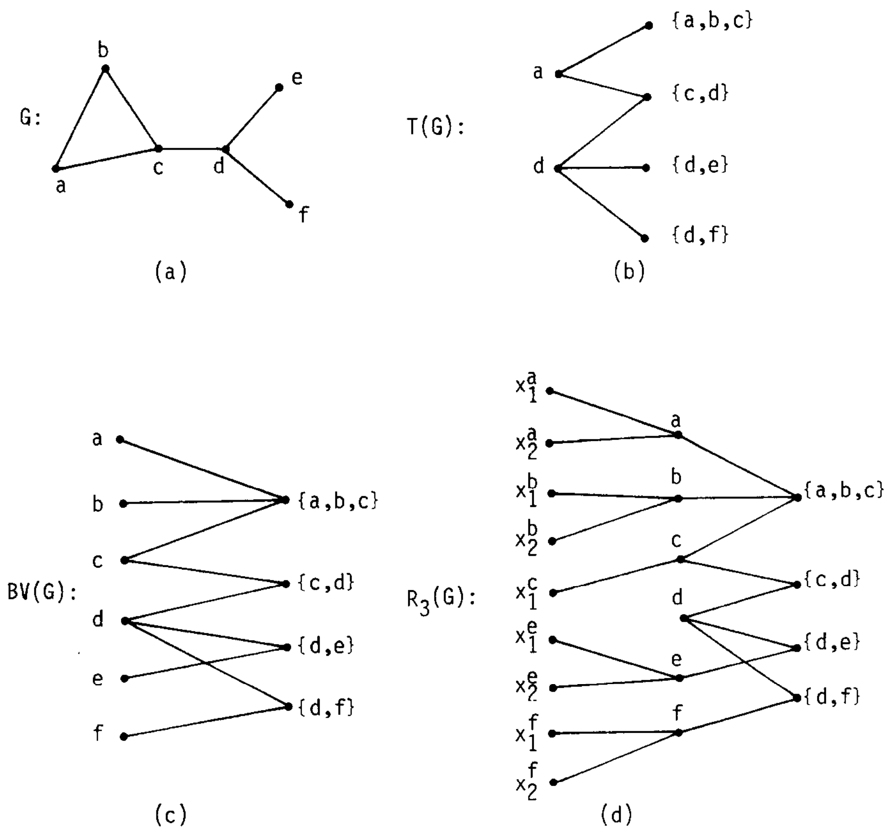


FIG. 4. A block graph, G, with its block-cutpoint tree, $T(G)$, its block-vertex-incidence tree, $BV(G)$, and its regularized-block-vertex-incidence tree, $R_3(G)$.

LEMMA 6.1. *If H is any block graph then $u \sim_H v$ if and only if $u \sim_{BV(H)} v$, for all $u, v \in V(H)$.*

Proof. Any automorphism of H extends to an automorphism of $BV(H)$ in the obvious way. For the converse, it suffices to note that any automorphism of $BV(H)$ fixes both V and B, and the restriction to V induces an automorphism of H. ∎

Unfortunately, $BV(G)$ does not always characterize G up to the removal of vertices, that is $G \backslash u \cong G \backslash v \not\Rightarrow BV(G) \backslash u \cong BV(G) \backslash v$. However, a further extension to $BV(G)$ provides a tree characterization with this additional property.

Let H be an arbitrary connected graph. We define the *cut-degree* of an arbitrary vertex v of H, denoted $\delta_c(v, H)$, to be the number of connected components of $H\backslash v$. (The cut degree of a vertex in an arbitrary graph is just the cut degree of that vertex in its connected component). Finally, the cut-degree of H, denoted $\delta_c(H)$, is given by $\delta_c(H) = \max_{v \in V(H)} \delta_c(v, H)$. Note that if H is a block graph then $\delta_c(v, H)$ is exactly the degree of v in $BV(H)$.

We now introduce what we call a *regularized-block-vertex-incidence-tree* of G. Let $t \geqslant \delta_c(G)$. The graph $R_t(G)$ is formed from $BV(G)$ by adding $t - \delta_c(v, G)$ new edges incident with each vertex $v \in V$. More formally, let $X = \{x_i^v \mid v \in V,\ 1 \leqslant i \leqslant t - \delta_c(v, G)\}$. Then $R_t(G)$ is the tree with vertex set $B \cup V \cup X$ and edge set

$$\{(b, v) \in B \times V \mid v \in b\} \cup \{(v, x_i^v) \mid v \in V,\ 1 \leqslant i \leqslant t - \delta_c(v, G)\}$$

(see Fig. 4d). As before, if H is an arbitrary block graph with connected components $H_1, \ldots, H_t$, then $R_t(H)$ is the forest $\bigcup_{i=1}^{t} R_t(H_i)$.

Note that if the elements of X are interpreted as pseudo-blocks of size one, then $R_t(G)$ can be interpreted as a block-cutpoint tree where every vertex of G belongs to exactly t blocks (including pseudo-blocks). The advantage of adding these pseudo-blocks is that the property of being a regularized-block-vertex-incidence-tree is preserved (ignoring isolated vertices) by the removal of V-vertices. Specifically,

LEMMA 6.2. *For all $v \in V$ and $t \geqslant \delta_c(G)$,*

$$R_t(G)\backslash v \cong R_t(G\backslash v) \cup (t - \delta_c(v, G)) \cdot K_1.$$

Proof. If $\delta_c(v, G) = 1$, the result follows directly from the definitions. The general result follows by induction on $\delta_c(v, G)$. ∎

We can now state more formally our claim that $R_t(H)$ characterizes the block graph H up to both similarity and removal-similarity of vertices.

THEOREM 6.3. *Let H be any block graph with vertices u and v, and let $t \geqslant \delta_c(H)$. Then,*

(i) *$u \sim_H v$ if and only if $u \sim_{R_t(H)} v$; and*

(ii) *$H\backslash u \cong H\backslash v$ if and only if $R_t(H)\backslash u \cong R_t(H)\backslash v$.*

Proof. Part (i) follows immediately from Lemma 6.1 and the observation that every automorphism of $R_t(H)$ must fix X (that is, map vertices of X onto vertices of X). Part (ii) is a direct consequence of Lemma 6.2. ∎

COROLLARY 6.4. *Vertices u and v are pseudosimilar in the block graph H if and only if u and v are pseudosimilar in the forest $R_t(H)$.*

It follows from Corollary 6.4 that results on pseudosimilarity in (connected) forests carry over directly to (connected) block graphs. For example, we can now deduce Theorem 4 of [4] in its full generality from our Corollary 3.3.

7. Edge Pseudosimilarity

We have to this point been discussing the similarity (or pseudosimilarity) of pairs of vertices in a tree. These notions have natural analogues for edges as well, as do questions regarding the relationship between similarity and removal-similarity [5]. Fortunately, it is not necessary to rederive all of our vertex-based results in order to establish the corresponding results for edge similarity.

Two edges x and y in a graph G are *similar*, denoted $x \sim_G y$ (or simply $x \sim y$ when G is clear from the context), if there exists an automorphism of G taking x onto y. To be consistent with our earlier notion we let $G \backslash x$ denote the graph with the edge x (but not its endpoints) removed. Two edges x and y satisfying $G \backslash x \cong G \backslash y$ are said to be *removal-similar in G*. As before edges which are removal-similar but not similar are said to be *pseudosimilar*.

Questions concerning edge similarity and pseudosimilarity are easily reduced to questions of vertex similarity and pseudosimilarity by means of the subdivision graph associated with a given tree. If $G = (V, E)$ is any graph then the *subdivision graph* of G, denoted $S(G)$, is the bipartite graph $(V \cup E, E')$ where $v \in V$ is joined to $e \in E$ to form an element $(v, e) \in E'$ exactly when v is an endpoint of e in G.

Proposition 7.1. *T is a tree if and only if $S(T)$ is a tree.*

Of particular importance for questions concerning edge similarity and pseudosimilarity in trees is the following lemma whose proof follows in a straightforward way from the above definitions.

Lemma 7.2. (a) *If T_1 and T_2 are trees then $S(T_1) \cong S(T_2)$ if and only if $T_1 \cong T_2$.*

(b) *If T is any tree and $e \in E(T)$ (and hence $e \in V(S(T))$), then $S(T \backslash e) \cong S(T) \backslash e$.*

The following theorem follows directly from the above definitions and Lemma 7.2.

Theorem 7.3. *If T is any tree with edges x and y, then*

(i) *$x \sim_T y$ if and only if $x \sim_{S(T)} y$; and*

(ii) *x and y are edge-removal-similar in T if and only if x and y are vertex-removal-similar in $S(T)$.*

It follows from Theorem 7.3 that edges x and y are pseudosimilar in T if and only if they are pseudosimilar (as vertices) in $S(T)$. Hence the characterization of Theorem 3.7 leads directly to a characterization of trees with pseudosimilar edges.

THEOREM 7.4. *If T is any tree with pseudosimilar edges x and y then for some integers $t > 1$ and $h > 2t$ there exist rooted trees Y_k, $1 \leqslant k \leqslant h$, where $Y_k \overset{*}{\cong} Y_{k+t}$, $1 \leqslant k \leqslant h-t$, such that $T = \langle Y_1, Y_2, \ldots, Y_h \rangle$, $x = (y_t, y_{t+1})$, $y = (y_{h-t}, y_{h-t+1})$, and $\langle Y_1, \ldots, Y_t \rangle \cong \langle Y_{h-t+1}, \ldots, Y_h \rangle$.*

Proof. This characterization is a direct consequence of the characterization of $S(T)$ given by Theorem 7.3. ∎

If x is any edge of the graph G then Γ_x denotes the set of edges (including x) that are incident on at least one endpoint of x (Γ_x^k could be defined analogously; cf. Section 1). Two edges x and y are full-1-removal-similar if $G \backslash x \cong G \backslash y$ and $G \backslash \Gamma_x \cong G \backslash \Gamma_y$. As with vertices, full-1-removal-similarity implies similarity of edges.

THEOREM 7.5. *If T is any tree with edges x and y, then $T \backslash x \cong T \backslash y$ and $T \backslash \Gamma_x \cong T \backslash \Gamma_y$ implies $x \sim_T y$.*

Proof. The argument parallels the proof of Theorem 5.1, with Theorem 3.7 replaced by Theorem 7.4. It is sufficient to look only at case ii of the proof. ∎

8. CONCLUSIONS

We have presented a new characterization of trees with pseudosimilar vertices. This characterization leads directly to related characterizations of forests and block-graphs with pseudosimilar vertices and of trees with pseudosimilar edges.

In addition, we have been able to conclude from our characterization that, unlike the situation for general graphs, in trees it is not possible to have three or more mutually pseudosimilar vertices, nor is it possible to have full-1-pseudosimilar vertices.

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